Interpolation

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Version 1.0

Interpolation allows the reconstruction of an analog signal from a sampled version.

Definition 0.1 (Interpolation)
Interpolation is an application from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$, which maps a digital signal x to a reconstructed analog signal x_r whose x is the result of sampling.

1 Ideal interpolation

Since sampling duplicates the spectrum of the original analog signal, places a copy in $n\omega_s$ for every $n \in \mathbb{Z}$, and divides its modulus by T_s, we can apply an ideal lowpass filter followed by an amplifier of factor T_s to only have the copy centered in 0 with the correct magnitude.
Consider an analog signal x with spectrum X which was sampled with period T_s to obtain the sampled signal x_s. The corresponding digital signal, also denoted x, is such that for any $n \in \mathbb{Z}$, $x[n] = x(nT_s)$. The cutoff frequency of the lowpass filter must be $\omega_{co} = \dfrac{\omega_e}{2}$. We have seen in the lecture about frequency design that the impulse response an ideal lowpass filter with cutoff frquency ω_{co} followed by an amplifier of factor T_s is:

$$\forall t \in \mathbb{R} \qquad h(t) = T_s \frac{\sin(\omega_{co} t)}{\pi t} = T_s \frac{\sin\left(\frac{\omega_s}{2} t\right)}{\pi t} = \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\frac{\pi t}{T_s}} = \operatorname{sinc}\left(\frac{\pi t}{T_s}\right)$$

The reconstructed signal x_r, output of this system, is then:

$$\begin{aligned}\forall t \in \mathbb{R} \qquad x_r(t) &= (x_s * h)(t) = (h * x_s)(t) = \left(h * \sum_{n=-\infty}^{+\infty} x(nT_s)\delta_{nT_s}\right)(t) \\ &= \sum_{n=-\infty}^{+\infty} x(nT_s)(h * \delta_{nT_s})(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)h(t - nT_s) = \sum_{n=-\infty}^{+\infty} x(nT_s) \operatorname{sinc}\left(\frac{\pi(t - nT_s)}{T_s}\right)\end{aligned}$$

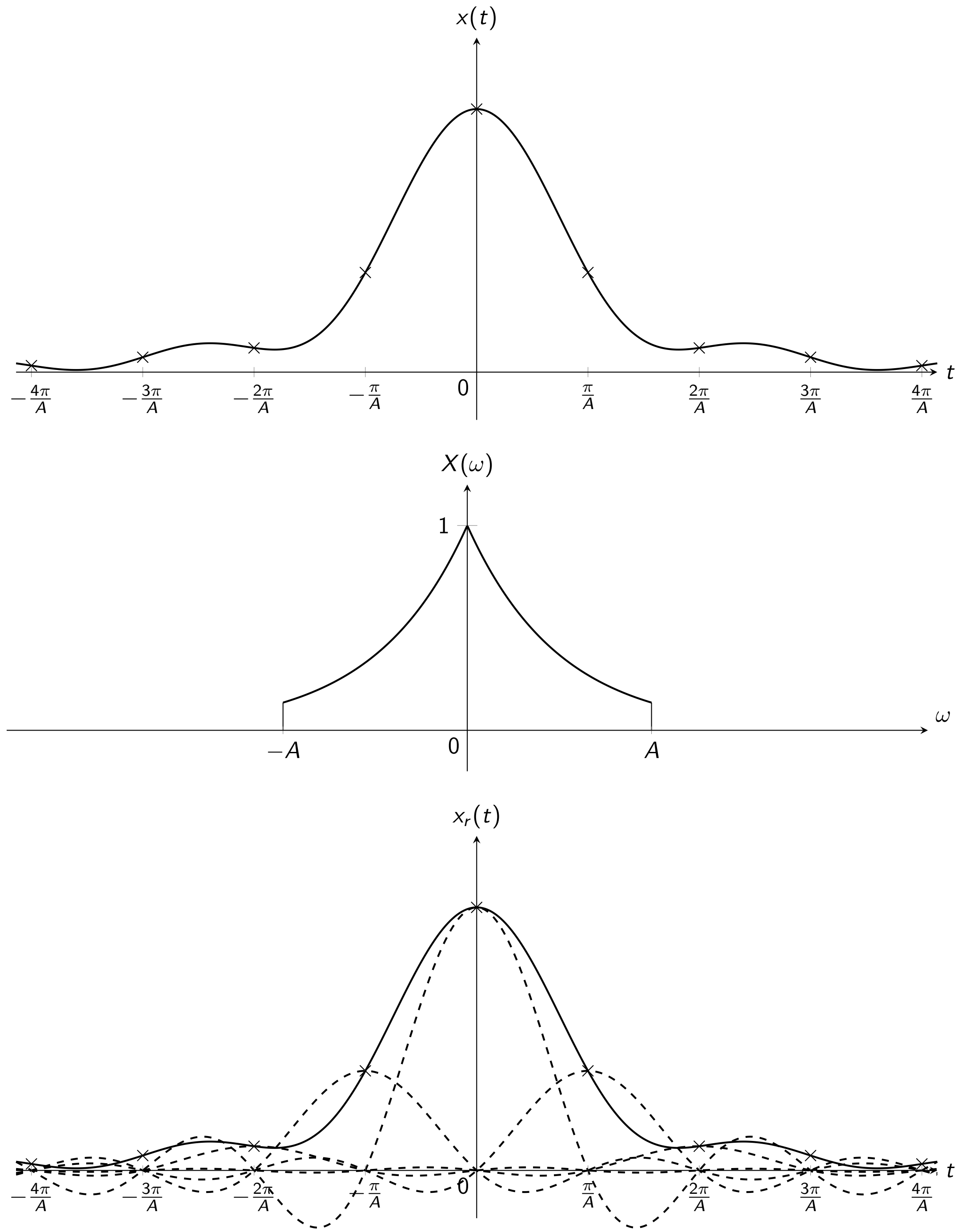
Example 1.1
Let a and A be two positive real numbers. Consider the analog signal:

$$\forall t \in \mathbb{R} \qquad x_A(t) = \frac{1}{\pi} \frac{a + e^{-aA}(t\sin(At) - a\cos(At))}{t^2 + a^2}$$

whose Fourier transform is the frequency-limited spectrum:

$$\forall \omega \in \mathbb{R} \qquad X_A(\omega) = R_{[-A,A]}(\omega) e^{-a|\omega|}$$

where A is the maximum frequency. We sample this signal with frequency $\omega_s = 2A$, i.e. $T_s = \dfrac{\pi}{A}$.



The sum of the dashed curves, each corresponding to a single shifted and weighted sinc function, provides the reconstructed signal x_r.

Remark: Not only this interpolation is not causal because it requires the knowledge of the future samples, but it also needs the infinity of all the samples at once. In the following sections, we introduce interpolation systems that are causal and implementable because they only require the local knowledge of past samples.

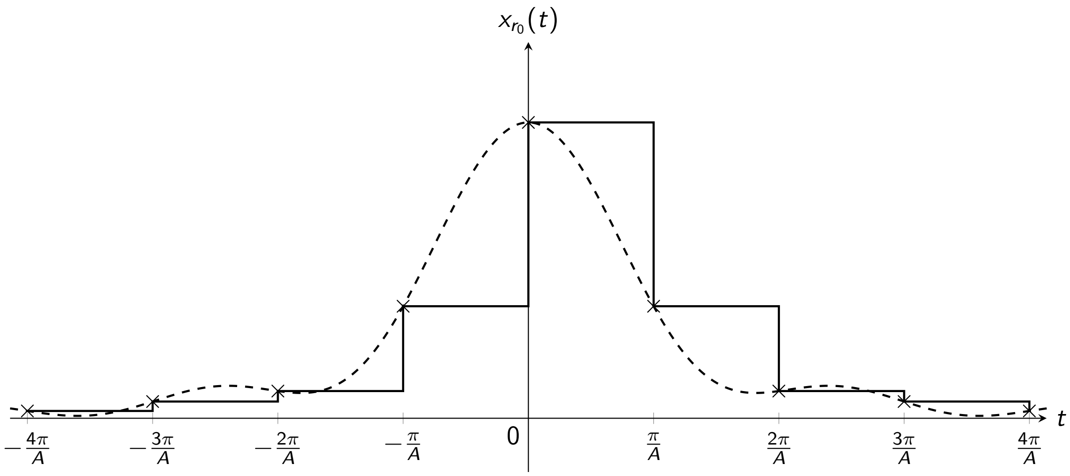
2 Zero-order hold

We start with the zero-order hold which generates a piecewise constant signal from the digital samples. We study the impulse response of this hold and its effects on the spectrum of the reconstructed signal.

Definition 2.1 (Zero-order hold)
The **zero-order hold** with period T_s is a system from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$ which maps a digital signal $(x[n])_{n\in\mathbb{Z}}$ to the reconstructed analog signal x_{r_0} defined by:

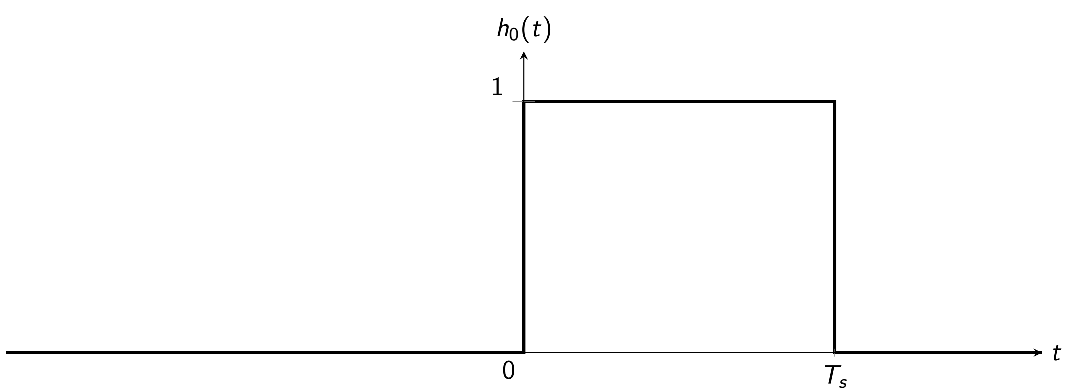
$$\forall n \in \mathbb{Z} \qquad \forall t \in [nT_s, (n+1)T_s[\qquad x_{r_0}(t) = x[n]$$

Example 2.1
For the analog signal of the previous example, we obtain the following reconstructed signal x_{r_0}:



Proposition 2.1
The impulse response h_0 of the zero-order hold with period T_s is defined over $\mathbb{R}$ by

$$h_0(t) = \begin{cases} 1 & \text{if } t \in [0,\, T_s[\\ 0 & \text{otherwise} \end{cases}$$



PROOF : We simply have to apply the definition of the zero-order hold to the digital Dirac delta function. ∎

With this impulse response, the reconstructed signal x_{r_0} of a digital signal x corresponding to the sampled signal x_s is:

$$\forall t \in \mathbb{R} \qquad x_{r_0}(t) = (x_s * h_0)(t) = \sum_{n=-\infty}^{+\infty} x[n]h_0(t - nT_s)$$

Proposition 2.2

The frequency response H_0 of the zero-order hold with period T_s is given by:

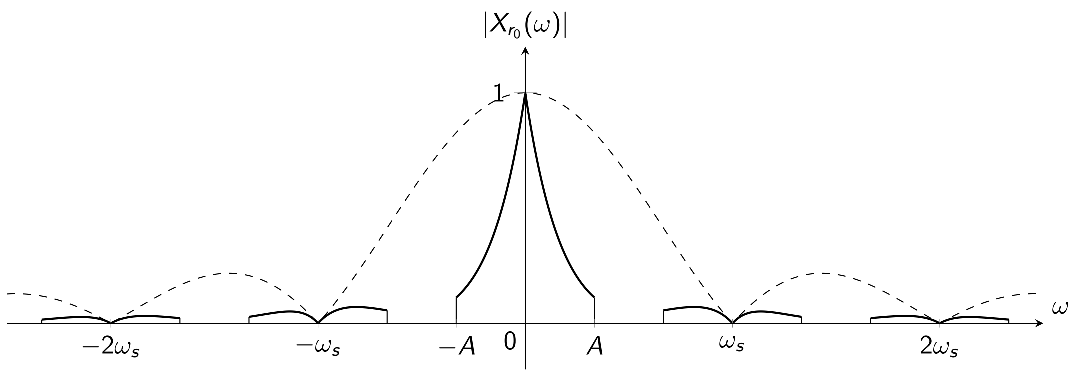
$$\forall \omega \in \mathbb{R} \qquad H_0(\omega) = T_s \operatorname{sinc}\left(\omega \frac{T_s}{2}\right) \exp\left(-i\omega \frac{T_s}{2}\right) = \frac{2}{\omega} \sin\left(\omega \frac{T_s}{2}\right) \exp\left(-i\omega \frac{T_s}{2}\right)$$

PROOF : The impulse response h_0 is a shifted version of the rectangle signal $R_{[-\frac{T_s}{2}, \frac{T_s}{2}]}$, i.e. $h_0 = \tau_{\frac{T_s}{2}}\left(R_{[-\frac{T_s}{2}, \frac{T_s}{2}]}\right)$. Thus we have, for any $\omega \in \mathbb{R}$,

$$H_0(\omega) = \mathcal{F}(h_0)(\omega) = \mathcal{F}(R_{[-\frac{T_s}{2}, \frac{T_e}{2}]})(\omega) \exp\left(-i\omega \frac{T_s}{2}\right) = T_s \operatorname{sinc}\left(\omega \frac{T_s}{2}\right) \exp\left(-i\omega \frac{T_s}{2}\right) \qquad \blacksquare$$

Example 2.2

The spectrum of the reconstructed signal for the previous example is then:



We see on this figure that the frequency response H_0 does not completely remove the undesired copies in the reconstructed spectrum. In addition, it shifts its phase because of the factor $\exp\left(-i\omega \frac{T_s}{2}\right)$.

3 First-order hold

A major flaw of the zero-order hold is that it generates a piecewise constant reconstructed signal with discontinuities in nT_s for all $n \in \mathbb{Z}$. To circumvent this issue, we can create a piecewise linear reconstructed signal by connecting the samples with straight lines.

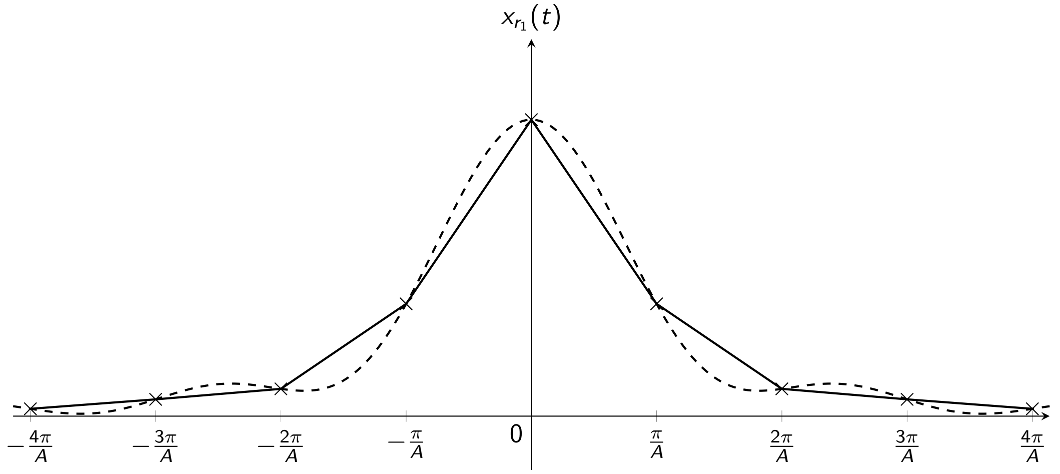
Definition 3.1 (Ideal first-order hold)

The **ideal first-order hold** with period T_s is the system $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$ which maps a digital signal $(x[n])_{n\in\mathbb{Z}}$ to the reconstructed analog signal x_{r_1} defined by

$$\forall n \in \mathbb{Z} \qquad \forall t \in [nT_s, (n+1)T_s[\qquad x_{r_1}(t) = \frac{x[n+1] - x[n]}{T_s}(t - nT_s) + x[n]$$

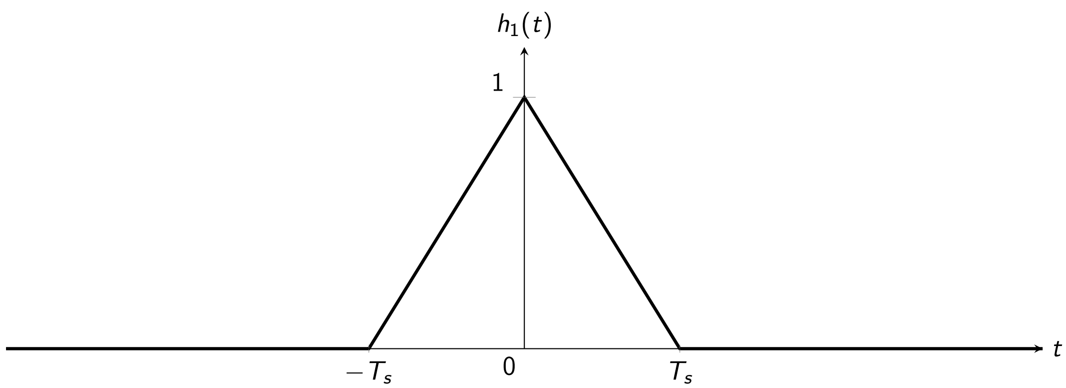
Example 3.1

The signal reconstructed by the ideal first-order signal for the previous example is:



Proposition 3.1

The impulse response of the first-order hold is the triangle signal $h_1 = \dfrac{1}{T_s} T_{T_s}$.



PROOF : We simply have to apply the definition of the ideal first-order hold to the digital Dirac delta function. ■

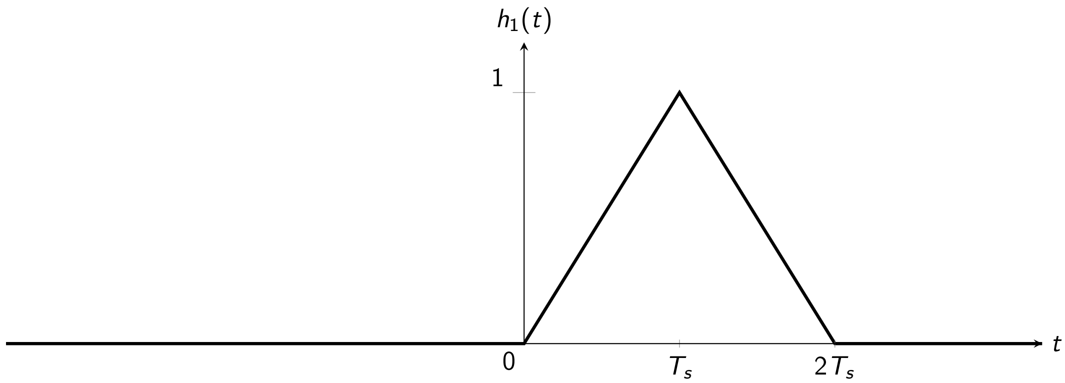
Remark: Since for any $t \in [-T_s, 0]$, $h_1(t) \neq 0$, the ideal first-order hold is not a causal system, thus it is not implementable. This fact is visible in the definition of the hold because generating x_{r_1} over $[nT_s, (n+1)T_s[$ requires the knowledge of sample $x[n+1] = x((n+1)T_s)$. To make this ideal system implementable, we need to shift its impulse response by a delay T_s.

Definition 3.2 (Causal first-order hold)

The **causal first-order hold** with period T_s is the system from $\mathcal{F}(\mathbb{Z}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$ which maps a digital $(x[n])_{n\in\mathbb{Z}}$ to the reconstructed analog signal x_{r_1} defined by:

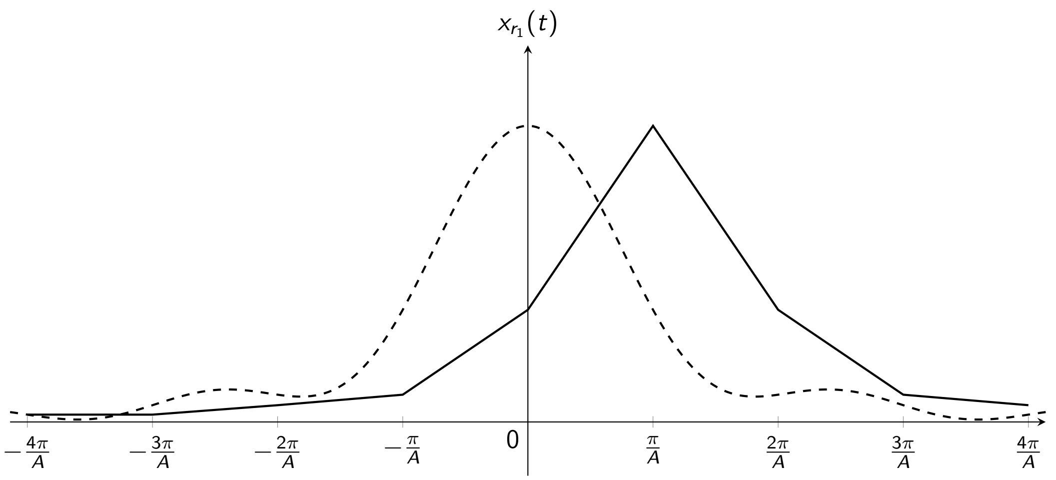
$$\forall n \in \mathbb{Z} \qquad \forall t \in [nT_s, (n+1)T_s[\qquad x_{r_1}(t) = \frac{x[n] - x[n-1]}{T_s}(t - nT_s) + x[n-1]$$

The impulse response of this hold is then $h_1 = \dfrac{1}{T_s} \tau_{T_s}(T_{T_s})$.



Example 3.2

The signal reconstructed by the causal first-order signal for the previous example is:



Proposition 3.2

The frequency response H_1 of the causal first-order hold with period T_s is given by:

$$H_1(\omega) = T_s \operatorname{sinc}^2\left(\frac{T_s\omega}{2}\right) \exp\left(-i\omega T_s\right)$$

Proof : We have seen in the lecture about Fourier transform that the spectrum of triangle signal T_a is

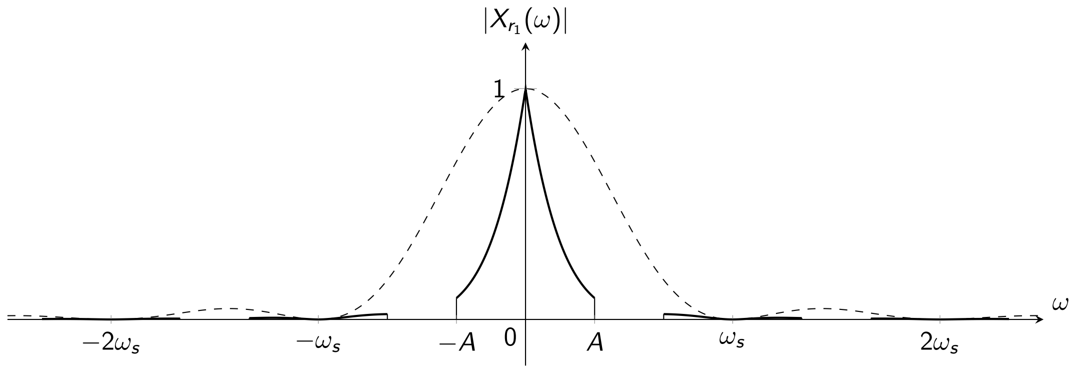
$$\mathcal{F}(T_a)(\omega) = a^2 \operatorname{sinc}^2\left(\frac{a\omega}{2}\right)$$

Using the linearity and time-shift properties of the Fourier transform, we get:

$$H_1(\omega) = T_s \operatorname{sinc}^2\left(\frac{T_s\omega}{2}\right) \exp\left(-i\omega T_s\right)$$ ∎

Example 3.3

The spectrum of the reconstructed signal for the previous example is then:



The causal first-order hold exhibits better results at eliminating the undesired high-frequency copies of the original spectrum but it leads to a doubled phase shift compared with the zero-order hold.